647510 **MSC2MatC203x** **Seat No :________**

M.Sc. (Maths) Semester - 2 (*CBCS*) Examination
May/June-2022 (NEW COURSE)
Topology – 2 (CORE)

Time: 2:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1 (A) Answer any Two out of Three. **(10)**

(1) Show that the topological space $\mathbb{R}_K$ is Hausdorff but not regular.

(2) Define T_1- space. Show that a topological space X is T_1 if and only for each pair of points of X, each has a neighborhood not containing the other.

(3) Let X and Y be topological spaces, and let Y be Hausdorff. If $f, g : X \to Y$ are continuous functions, then show that the set $\{x \in X: f(x) = g(x)\}$ is closed in X.

Q.1 (B) Answer any One out of Two. **(04)**

(1) Show that a topological space X is Hausdorff if and only if $\{(x, x): x \in X\}$ is a closed subset of the product space $X \times X$.

(2) Let X and Y be topological spaces, and let $p: X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is a compact subset of X for every $y \in Y$. If X is a Hausdorff space, then show that Y is a Hausdorff space.

Q.2 (A) Answer any Two out of Three. **(10)**

(1) Show that every regular space with countable basis is normal.

(2) Show that every locally compact Hausdorff space is regular.

(3) Let (X, d) be a metric space, and let A and B be disjoint closed subsets of X. Show that there exists a continuous function $f: X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$..

Q.2 (B) Answer any One out of Two. **(04)**

(1) Show that the product of completely regular spaces is completely regular.

(2) Show that every compact Hausdorff space is normal.

Q.3 (A) Answer any Two out of Three. **(10)**

(1) Let X and Y be topological spaces, let Y be compact, and let $x_0 \in X$. If N is an open subset of $X \times Y$ containing the slice $\{x_0\} \times Y$, then show that there is a neighbourhood W of x_0 in X such that $\{x_0\} \times Y \subset W \times Y \subset N$.

(2) Let X be a Hausdorff space. Show that X is locally compact if and only if given $x \in X$, given a neighbourhood U of x, there is a neighbourhood V of x such that $\bar{V} \subset U$ and $\bar{V}$ is compact.

(3) Show that the set of rationals $\mathbb{Q}$ with the usual topology is not locally compact.

Q.3 (B) Answer any One out of Two. **(04)**

(1) Show that $(0,1]$ is not limit point compact as a subspace of $\mathbb{R}_\ell$.

(2) Let X be a locally compact space, and let Y be a topological space. If $f: X \to Y$ is continuous, does it follow that $f(X)$ is locally compact? Justify your answer.

Q.4 (A) **Answer any Two out of Three.** **(10)**

(1) Let (X, d) be a metric space. If X is sequentially compact, then show that X is compact.

(2) Let (X, d) be a compact metric space. Let $f: X \to X$ satisfy $d\big(f(x), f(y)\big) = d(x, y)$ for all $x, y \in X$. Show that the map f is onto.

(3) Let A be an open covering of the metric space (X, d). If X is compact, then show that there is $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of A containing it.

Q.4 (B) **Answer any One out of Two.** **(04)**

(1) Let (X, d_X) and (Y, d_Y) be metric spaces, and let X be compact. If $f: X \to Y$ is continuous, then show that f is uniformly continuous.

(2) Let (X, d) be a metric space, let K be a compact subset of X, and let A be a subset of X. Show that there is $k \in K$ such that $d(k, A) = \inf\{d(x, y): x \in K, y \in A\}$.

Q.5 (A) **Answer any Two out of Three.** **(10)**

(1) Let (X, d) be a metric space. Show that there exists an embedding of X into a complete metric space.

(2) Let X be a topological space, and let (Y, d) be a complete metric. Show that the set $C(X, Y)$ of all bounded continuous functions from X to Y is a complete metric space under the uniform norm.

(3) Let (X, d_X) be a metric space, (Y, d_Y) be a complete metric space, and let $A \subset X$. If $f: A \to Y$ is uniformly continuous, then show that f can be uniquely extended to a uniformly continuous function $g: \bar{A} \to Y$.

Q.5 (B) **Answer any One out of Two.** **(04)**

(1) Let (X, d) be a metric space. If every Cauchy sequence in X has a convergent subsequence, then show that X is complete.

(2) Let (X, d) be a complete metric space. If (A_n) is a decreasing sequence of closed sets and if $diam(A_n) \to 0$ as $n \to \infty$, then show that $\bigcap_{n=1}^{\infty} A_n$ is nonempty.

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